Optimal Credit Limit
Management

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Introduction

A. Background

Main Applications of Credit Risk Models:

- Pricing of credit risky securities:
  - Corporate bonds, swaps, and vulnerable securities.
  - Collateralized bond obligations (CDOs’), basket credit derivatives (ith-to-default swap).

- Credit risk management:
  - Computation of loss distribution and associated risk measures (such as VaR, coherent measures, ...) for portfolios of defaultable bonds and loans.
  - Determination of risk capital.
B. Motivation

What happens during market downturns?

- Credit derivatives become very costly.
- Credit portfolio diversification cannot be pursued to the extend required.
- Banks stick to short-term policies to control credit exposure.

How to achieve a short-term control?

- Banks decrease the credit limits for their debtors.
- Examples are numerous: Swiss (UBS, Credit Suisse), ANZ ”de-risking” strategy for corporate debt portfolios.

⇒ Our goal is to analyze the optimal credit limit policy.
C. Preview

• Credit risk model as a two players game (bank and firm) in continuous time to obtain and analyze *optimal limit policy*:
  
  – closely linked to structural models, but with *endogenous firm value*;

• We develop the model in three steps:
  
  – **Model for Non-Defaultable Firms**
    
    * We analyze the risk/return profile, the optimal credit usage, and the optimal limit policy.
  
  – **Model with Partial Information**
    
    * We analyze the effect of the volatility of the signal.
    * Adverse selection and pooling.
  
  – **Contracting under Partial and Incomplete Information**
    
    * Removal of pooling effects,
    * Design of optimal contract,
    * Why some banks attract “bad debtors”.
Model Basics

A. The Economy

- Financial market defined on \((\Omega, \mathcal{F}, \mathbb{P})\) with a terminal time \(T\).
- Uncertainty is driven by a one dimensional Brownian motion \(\{W_t, \mathcal{F}_t; 0 \leq t \leq T\}\).
- The economy is populated by two agents \(B\) and \(C\).
- Agent \(C\) demands credit, agent \(B\) supplies credit. Therefore, we call \(C\) “a company”, and agent \(B\) “a bank”.
- Both agents optimize their value function, taking into account the optimizing behavior of the other agent.
B. Agent $C$ - The Company

- The company maximizes her final surplus,
  \[ S_T = A_T - L_T. \]

- Surplus dynamics
  \[ dS_t = \mu_0 dt + \sigma_0 dW_t, \quad S_0 \geq 0. \]

- The company can demand a credit amount $c_t S_t$, provided by agent $B$.

- The demand $c_t S_t$ is limited by $\ell_t$, the credit limit imposed by bank $B$.

- This additional amount of money can be invested such that

  \[ dS_t = (\mu_0 + (\mu_1 - p)c_t) dt + (\sigma_0 + \sigma_1 c_t) dW_t, \]

  where
  
  $c_t$ : credit demand as a fraction of surplus $S_t$;
  $p$ : cost of credit;
  $\mu_1$ : mean of new investment;
  $\sigma_1$ : volatility of new investment.
Agent $C'$ solves

$$
V(S, t) = \max_{c_t} \mathbb{E} \left[ \int_t^T e^{-\delta(T-s)} S_s ds \mid \mathcal{F}_t \right],
$$

s.t. $\text{var} \left[ \int_t^T S_s ds \mid \mathcal{F}_t \right] \leq \varsigma^2,$

$c_t S_t \leq \ell_t, \ \forall t \in [0, T],$

$0 \leq c_t, \ \forall t \in [0, T],$

$dS_t = (\mu_0 + (\mu_1 - p)c_t) dt + (\sigma_0 + \sigma_1 c_t) dW_t.$

$\Rightarrow (C')$ is not separable in the dynamic programming sense.

$\Rightarrow$ Embedding technique (Li and Ng (2000); Leippold, Trojani, and Vanini (2001)) gives a separable problem:

$$(C1): \quad V(S, t, \omega, \lambda) = \max_{c_t \in C(S)} \mathbb{E} \left[ \int_t^T e^{-\delta(T-s)} \left( \lambda S_s - \omega S_s^2 \right) ds \mid \mathcal{F}_t \right].$$

$\Rightarrow$ Choosing $\omega, \lambda$ appropriately, then, if $c^*$ solves $(C1)$, it also solves $(C')$. 

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C. Agent $B$ - The Bank

- The bank maximizes expected earnings minus capital costs from her credit lending business and solves

\[
(B1): \quad J(S, t) = \max_{\ell_t} \mathbb{E} \left[ \int_{t}^{T} e^{-\delta(T-s)} \left( p c_s S_s - \kappa \ell_s \right) \, ds \mid \mathcal{F}_t \right]
\]

\[
s.t. \quad dS_t = (\mu_0 + (\mu_1 - p)c_t) dt + (\sigma_0 + \sigma_1 c_t) dW_t,
\]

\[
\ell_t \geq 0, \quad \forall t \in [0, T].
\]

Hence, the bank optimizes the limit amount $\ell$ over the duration $T - t$.

- Following regulatory practice, the capital costs $\kappa$ are calculated on the limit amount set-off for the client and not on the actual credit exposure!
D. Problem Characteristics

- Problems (C1) and (B1) define a non-cooperative game, where the decision of each player affects the value function of the other one.

- Solution Concept:
  - Subgame-perfect Nash equilibrium (SPNE):
    A pair \((c^*, \ell^*)\) is a Nash equilibrium if
    \[
    J(S, t, c^*, \ell^*) \geq J(S, t, c^*, \ell) \quad \text{and} \quad V(S, t, c^*, \ell^*) \geq V(S, t, c, \ell^*)
    \]
    for any feasible policies \(c\) and \(\ell\) and where \(c^*\) (\(\ell^*\)) is the optimal strategy of \(C\) (\(B\)).
  - \((c^*, \ell^*)\) depend on the state \(S\) and time \(t\).
A. The Result

**Proposition 1.** Consider an economy with two players solving (C1) and (B1), respectively. The strategies

\[ c_t^* = \max \left( \frac{\ell_t^*}{S_t}, 0 \right), \quad \ell_t^* = \max \left( \gamma_1 S_t + \gamma_2 S_t^2, 0 \right), \]

are a subgame perfect Nash equilibrium, where \( \gamma_1 \) and \( \gamma_2 \) are constants.

The value functions read

\[
J(S, t) = e^{-\delta(T-t)} (p - \kappa) \left( b_0 + b_1 S_t + b_2 S_t^2 \right),
\]

\[
V(S, t) = e^{-\delta(T-t)} \left( k_0 + k_1 S_t + k_2 S_t^2 \right),
\]

Note: The parameter \( \kappa \) (\( = \) costs for providing limit) does not enter the optimal limit policy.
B. Discussion “Non-defaultable Firm”

• Critical level of financing $S_c = \frac{\delta(\mu_1 - p + \sqrt{\delta} \sigma_1)}{\delta \sigma_0 \sigma_1 - \mu_0 (\mu_1 - p)}$:

\[
S_c \begin{cases} 
< 0, & \text{if } \mu_0 > \frac{\delta \sigma_0 \sigma_1}{\mu_1 - p} > 0, \\
> 0, & \text{if } 0 < \mu_0 < \frac{\delta \sigma_0 \sigma_1}{\mu_1 - p},
\end{cases}
\]  

(1)

Thus, $S_c > 0$ becomes more likely, when

a) the difference between the mean of the investment opportunity $\mu_1$ and the costs of the credit becomes small,

b) the volatility of both investments, $\sigma_0$ and $\sigma_1$, are large,

c) the time preference parameter $\delta$ is large.

• The surplus is convex (concave) in $\ell^*$, if $\gamma_2 > 0$ ($\gamma_2 < 0$).

• The surplus dynamics follows a stationary process if and only if

$$(\mu_1 - p)\gamma_2 > 0.$$
Figure 1: Optimal limit policy.
Model with Partial Information

A. Problem Motivation

- Bank $B$ has only partial information about the true surplus of company $C$ (Enron, WorldCom, Swissair).
- $B$ makes a “best guess” $\hat{S}$ of the company’s surplus.
- The guess might be based on the credit usage $c_t$.
- The signal $\zeta_t = S_t + "noise"$ follows
  \[
  (O) \quad : \quad d\zeta_t = (A_0 + A_1 S_t) \, dt + B \, dZ_t.
  \]
- The Brownian motions $W_t$ and $Z_t$ are independent, $\hat{S}_t$ is $\mathcal{G}_t$-measurable, where
  \[
  \mathcal{G}_t = \sigma \{ Z_s : 0 \leq s \leq t \}.
  \]
- Again, the firm is non-defaultable.
B. The Company’s Decision

- $C$ knows that bank $B$ obtains a noisy signal of the current surplus. $C$ solves

\[(C2) : \quad V(S,t,\omega,\lambda) = \max_{c_t \in \mathcal{C}(\hat{S}_t)} \mathbb{E}\left[ \int_t^T \left( e^{-\delta(T-s)} \lambda S_s - \omega S_s^2 \right) ds \middle| \mathcal{F}_t \right].\]

- The optimal $c_t$ is given by

$$c_t^* = \max \left( \frac{\ell_t^*}{\hat{S}_t}, 0 \right).$$

- The state variable $S_t$ evolves according to

\[(U) : \quad dS_t = (\mu_0 + (\mu_1 - p)c_t^*) dt + (\sigma_0 + \sigma_1 c_t^*) dW_t.\]

- Note that:
  - we replaced $\mathcal{C}(S_t)$ by $\mathcal{C}(\hat{S}_t)$,
  - the conditional distribution $F_{c_0^*} = \mathbb{P}(S_t \leq k|\mathcal{G}_t)$ is ($\mathbb{P}$-a.s.) Gaussian.
C. The Bank’s Decision using Kalman Filter

- Bank obtains signal $\zeta_t$ and makes a best guess

$$\int_{\Omega} |S_t - \hat{S}_t|^2 d\mathbb{P} = \mathbb{E} \left[ |S_t - \hat{S}_t|^2 \right] = \inf_{Y} \left\{ \mathbb{E} \left[ |\hat{S}_t - Y|^2 \right] | Y \in L^2(\mathcal{G}, \mathbb{P}) \right\},$$

with $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space for $(W_t, Z_t)$ and $\mathbb{E}(\cdot)$ denotes the expectation w.r.t. $\mathbb{P}$.

**Proposition 2.** Given equations (U) and (O), the process $\hat{S}_t$ follows

$$d\hat{S}_t = \left( \mu_0 + (\mu_1 - p)c_t + \rho_t \frac{A^2_1}{B^2}(S_t - \hat{S}_t) \right) dt + \rho_t \frac{A_1}{B} dZ_t, \quad \hat{S}_0 = \mathbb{E} [S_0],$$

where $\rho_t$ solves a Riccati equation.

$\Rightarrow$ $\hat{S}$ dynamics are nonlinear.

$\Rightarrow$ We use perturbation theory to maintain analyycity of the bank’s value function $J(\hat{S})$. 
Perturbation Step 1: Linearize $\rho_t$

We use a first-order approximation $\rho^{(1)}_t$, such that $\rho_t = \rho^{(1)}_t + O\left((c_t \sigma_1)^2\right)$, i.e.,

$$\rho^{(1)}_t = r_0(t) + r_1(t)c_t.$$ 

Perturbation Step 2: $J$-function

• The optimization problem for the bank reads

$$(B2): \quad J(\hat{S}, t) = \max_{\ell_t \in \mathcal{B}(\hat{S})} \mathbb{E}\left[ \int_t^T e^{-\delta(T-s)} \left(p_{cs}\hat{S}_s - \kappa \ell_s\right) ds \mid \mathcal{G}_t \right].$$

• The HJB for $(B2)$ cannot be solved in closed form.

• We assume that $B$ is competent enough such that $\frac{S_t - \hat{S}_t}{S_t}$ is close to 0.

• Then, we expand the $J$-function around $\frac{S_t - \hat{S}_t}{S_t} \sim 0$. 
D. The Result

**Proposition 3.** Consider an economy with two players solving (B2) and (C2), where player B has only partial information on C’s surplus. Then, the strategies

\[
\ell_t^{(1)*} = \max \left( \hat{\gamma}_1(t) \hat{S}_t + \hat{\gamma}_2(t) \hat{S}_t^2, 0 \right) + O \left( \frac{S_t - \hat{S}_t}{S_t} \right),
\]

\[
c_t^{(1)*} = \max \left( \ell_t^{*} \frac{0}{\hat{S}_t}, 0 \right) + O \left( \frac{S_t - \hat{S}_t}{S_t} \right),
\]

are a subgame perfect Nash equilibrium.

We note:

- The value function of B and C are again quadratic, but in \( \hat{S} \).
- Partial information introduces time-dependent parameters.
- Differences in optimal policies given by changes in the volatility terms:
  \[
  \sigma_0 \to r_0(t) \frac{A_1}{B}, \quad \sigma_1 \to r_1(t) \frac{A_1}{B}.
  \]
- Already to first-order, the differences in the models are significant!
Partial Information and Adverse Selection

Figure 2: The influence of signal variance for low \( \mu_1 (\mu_1 = 5\%) \). We make the following assumptions: \( \mu_0 = 2\% \), \( \sigma_0 = 10\% \), \( \sigma_1 = 30\% \), \( \delta = 5\% \), \( p = 1\% \). For the signal process \( \zeta_t \) we assume \( A_1 = -0.2 \). As ‘low signal variance’ we choose \( B = 7.5\% \), for ‘medium signal variance’ \( B = 20\% \), and for ‘high signal variance’ \( B = 50\% \). The bold dashed line is the satiation level. The thin dashed line denotes the optimal limit policy with full information. Finally, the bold straight line is the optimal limit policy under partial information.
Figure 3: The influence of signal variance for high $\mu_1$ ($\mu_1 = 10\%$). We make the following assumptions: $\mu_0 = 2\%$, $\sigma_0 = 10\%$, $\sigma_1 = 30\%$, $\delta = 5\%$, $p = 1\%$. For the signal process $\zeta_t$ we assume $A_1 = -0.2$. As ‘low signal variance’ we choose $B = 7.5\%$, for ‘medium signal variance’ $B = 20\%$, and for ‘high signal variance’ $B = 50\%$. The bold dashed line is the satiation level. The thin dashed line denotes the optimal limit policy with full information. Finally, the bold straight line is the optimal limit policy under partial information.
Optimal Contracting

A. Problem Motivation

- Partial information introduces adverse selection effects.
- Optimal limit policy for high and low signal variances is possibly reduced considerably, compared to case with full information.
- Low surplus reduces bank’s utility from providing credit.
- Firm has vital interest in obtaining large limits.
B. Basic Mechanisms

Firm:

- Has to undertake some costly effort to diminish noise acting on true state.
- Reducing noise reduces signal variance.

Bank:

- Cannot observe the firm’s effort.
- To differentiate between high and low efforts, bank sets up a contract:
  - Bank rewards efforts by using a compensation scheme $K$.
  - Contract needs to be incentive compatible for the firm and be at least as good as the next best opportunity.

Credit demand $c(\varepsilon)$ and the payment $K(\varepsilon)$ has to satisfy the following requirements:

1. Incentive constraint (IC).
2. Rationality constraint (PC).
C. The Bank’s Decision

The primary formal model with incomplete and partial information reads:

\[ J(\hat{S}, t) = \max_{\ell \in \mathcal{B}, K \in \mathcal{K}} \mathbb{E} \left[ \int_t^T e^{-\delta(T-s)} \left( p\hat{c}_s\hat{S}_s(\varepsilon) - \kappa\ell_s - K(\varepsilon) \right) ds \big| G_t \right]. \]

subject to

\[ \varepsilon^*, c^*_t \in \arg\max_{c_t \in \mathcal{C}, \varepsilon} \mathbb{E} \left[ \int_t^T e^{-\delta(T-s)} \left( \lambda S_s(\varepsilon) - \omega S^2_s(\varepsilon) + K(\varepsilon) \right) ds \big| \mathcal{F}_t \right], \quad (IC) \]

\[ \bar{u} \leq \mathbb{E} \left[ \int_t^T e^{-\delta(T-s)} \left( \lambda S_s(\varepsilon) - \omega S^2_s(\varepsilon) + K(\varepsilon) \right) ds \big| \mathcal{F}_t \right], \quad (PC) \]

\[ dS_t = (\mu_0 + (\mu_1 - (p + \varepsilon))c_t)dt + (\sigma_0 + \sigma_1c_t)dZ_t. \]

The program \((\mathcal{B}3)\) generalizes the standard theory of contracts with hidden information:

1. \((\mathcal{B}3)\) is a dynamic program.

2. The state variable \(S\) is not observable for the bank. Instead, there is a noisy signal. This defines partial information about the state variable.
D. Solution

**Proposition 4.** With $J^C$ and $J^B$ the value functions of the firm and bank, the optimality conditions are:

$$\hat{A}_{\hat{S}}J^B + \frac{\partial J^B(\hat{S})}{\partial \hat{c}} + \frac{\partial J^C(\hat{S})}{\partial \hat{c}} = \frac{1 - F(\epsilon) \partial^2 u_1(y(\epsilon))}{f(\epsilon) \partial \epsilon \partial \hat{c}} + \Delta \hat{A}_{\hat{S}}J^B$$

where $A_{\hat{S}}$ is the generator of the filter state dynamics $\hat{S}$ without incomplete information, $\Delta A_{\hat{S}} = \epsilon \hat{c} \frac{\partial}{\partial \hat{c}}$ is the generator corrections due to incomplete information.

Summarizing:

<table>
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<th>information</th>
<th>complete/partial</th>
<th>incomplete/partial</th>
</tr>
</thead>
<tbody>
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<td>static</td>
<td>(A)</td>
<td>(A) + (C)</td>
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<tr>
<td>dynamic</td>
<td>(A) + (B_1)</td>
<td>(A) + (B_1) + (C) + (B_2)</td>
</tr>
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E. Interpretation

⇝ Bank faces tradeoff between maximizing surplus \((A)\) and appropriating the firm’s information rent \((C)\).

⇝ In a static economy, the optimum is obtained when an increase in surplus equals the expected increase in agent C’s rate.

⇝ When \(\epsilon = \bar{\epsilon}\), \((\bar{C}) = 0\) and the company’s effort is of no concern. Only \((A)\) is maximized (“no distortion at the top”).

⇝ Within a dynamic setup, two additional terms appear:

\(B_1\) : The generator of the filter state dynamics \(\hat{S}\) without incomplete information

\(B_2\) : A dynamic hedging component against incomplete information on \(\epsilon\).
(cont’d)

The impact of \( (C') \) on the bank’s marginal utility function depends on the sign of the cross-derivative term \( \frac{\partial^2 u_1(y(\varepsilon))}{\partial \varepsilon \partial \hat{c}} \). If the effort \( \varepsilon \) and the credit usage \( c \) are

- substitutes, \( (C') > 0 \). The information rent to be payed by the bank is decreasing, when the effort \( \varepsilon \) is increasing. High efforts imply a low information rent. As the information rent is negative, high efforts increase the bank’s utility.

- complements, \( (C') < 0 \). The bank obtains a positive information rent. This information rent will become even more positive and inducing a higher utility for the bank if the firm’s effort is decreased. Hence, the bank has an interest to attract “bad debtors.”
Summary

- Endogenous firm values can be analyzed in closed-form using a subgame perfect Nash equilibrium.
- Possibility of credit usage may alter statistical properties of surplus dynamics.
- Partial information leads to adverse selection/pooling.
- Design of optimal contract.
- Rationalizing “bad” debt.

...proposal for future research:

1. Multiple companies demanding credit.
2. Allowing for default.
3. Embedding credit rating.